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Models for Effective Deployment and Redistribution of Bicycles Within Public Bicycle-Sharing Systems

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We develop practical operations research models to support decision making in the design and management of public bicycle-sharing systems. We develop a network flow model with proportionality constraints to estimate the flow of bicycles within the network and the number of trips supported, given an initial allocation of bicycles at each station. We also examine the effectiveness of periodic redistribution of bicycles in the network to support greater flow, and the impact on the number of docks needed.

We conduct our numerical analysis using transit data from train operators in Singapore. Given that a substantial proportion of passengers in the train system commute a short distance—more than 16% of passengers alight within two stops from the origin—this forms a latent segment of demand for a bicycle-sharing program. We argue that for a bicycle-sharing system to be most effective for this customer segment, the system must deploy the right number of bicycles at the right places, because this affects the utilization rate of the bicycles and how bicycles circulate within the system. We also identify the appropriate operational environments in which periodic redistribution of bicycles will be most effective for improving system performance.

1. Introduction

With heightened concerns about global oil prices, carbon emissions, and traffic congestion, governments around the world are exploring ways to "nudge" urban residents to commute using public transport instead of private automobiles. Several cities have set up public bicycle-sharing systems to facilitate short trips within the city. A bicycle-sharing system (BSS) is "a self-service short term, one-way-capable, bike rental offer in public spaces, with network characteristics" (OBIS Project 2011, p. 10). A standard BSS consists of a network of bicycle stations where bicycles are docked and available for pick up. Licensed operators often use low price or even free access to bicycles (for limited time) to entice commuters to adopt this transport mode. As of April 2013, there were around 535 bicycle-sharing programs around the world, with an estimated fleet of 517,000 bicycles (http://en.wikipedia.org/wiki/Bicycle_sharing_system).

    The advantages of using bicycle sharing include increased transit use, decreased personal vehicle trips, lower greenhouse gas emission, and improved public health. DeMaio (2009, p. 52) concluded that "as the price of fuel rises, traffic congestion worsens, populations grow, and a greater worldwide consciousness arises around climate change, it will be necessary for leaders around the world to find new modes of transport and better adapt existing modes to move people in more environmentally sound, efficient, and economically feasible ways. Bicycle sharing is evolving rapidly to fit the needs of the 21st century."

    Several cities in China have already started public bicycle projects, with Hangzhou now running arguably the world's largest bicycle-sharing program, with 50,000 bicycles deployed across 2,000 stations. It has close to 1.2 million registered users. This dwarfs the more famous VELIB program in France, which has around 20,000 bicycles deployed across 1,451 stations. In Kaohsiung, Taiwan, the first BSS, called C-Bike, was initiated in late February 2009. It originally had 20 bicycle stations, all located near train stations, with 1,500 bicycles deployed. By May 2009, it had 30 more bicycle stations in scenic areas, business

districts, government buildings, schools, etc., with a total of 4,500 bicycles in operation. Most stations are automatic, unmanned, self-help, and open 24 hours a day. One can check out or return a bicycle, using a credit card, in seconds. The proximity of bicycle stations has boosted use of C-Bikes. Total riding time for C-Bikes increased from 5,433 hours in April 2009 to 30,000 hours in December 2009. By the end of 2009, C-Bike had purportedly helped to reduce more than 200 tons of carbon-dioxide emissions in the city.

Although the BSS is very attractive as an alternative form of transportation, major challenges confront the operators and few scientific tools are available to support BSS design and effective management to enhance its economic viability. It is difficult for a BSS operator to turn a profit based on revenues collected on bicycle rides. In fact, to the best of our knowledge, none of the existing BSSs in operation has turned a profit. Most BSSs currently rely on government subsidies or private donations to sustain their operations.

One way to address the financial concern is through business-model innovation. In recent years, new BSS vendors have emerged with proprietary systems that they then sold to other BSS operators around the world. This allows system interoperability, and users can gain access to BSSs in multiple regions. Also, start-ups like CityRide are converting bike rides into carbon offset that can be sold on the carbon market. This evolution in business and pricing strategies has allowed BSSs to seek out business models that may be profitable, and thus ensure that new BSSs will continue to be set up around the world, regardless of their goals or scale. In this paper, we focus primarily on developing operations research (OR) tools to enhance the economic viability of the BSS by optimizing bicycle deployment and redistribution operations.

1.1. Related Literature

Although the first BSS was founded in Amsterdam on July 28, 1965, and the third generation of BSSs is now widely used around the globe, study of the design and management of BSSs is limited and only began recently; a few papers have focused on the history and real-life application of BSSs. We refer interested readers to DeMaio (2003, 2009), DeMaio and Gifford (2004), Lathia et al. (2012), and the references therein. We focus here mainly on reviewing models for bicycle-sharing-system design and bicycle redistribution.

Lin and Yang (2011) and Lin et al. (2013) study the design of public bicycle networks using the notion of service level constraints that are well grounded in the area of logistics and inventory management. Unfortunately, these studies fail to take into account the fact that whereas the flow of materials within a traditional logistics network is largely dictated and optimized by supply chain planners, the flow of material within a public bicycle system is dictated by the random travel patterns of passengers. Raviv and Kolka (2013) propose an inventory model to study

the management of bicycle stations in BSSs. Their study is based on a single station, which can inform decision making on docking capacity and bicycle redistribution (see also Raviv et al. 2012). Nair and Miller-Hooks (2011) develop a stochastic joint chance constraint model to study vehicle redistribution in vehicle-sharing systems. Schuijbroek et al. (2013) further develop a model to construct vehicle routes for bicycle redistribution.

1.2. Research Issues and Structure of the Paper

Given the success of such programs in the United States, where the biking population is not even 1%, the number of people already using bicycles is not considered essential. Coverage and density of bicycle stations are, in contrast, critical: stations should be separated by not more than 500 m, and ideally not more than 300 m. However, excessive focus on coverage and density, without adequate understanding of the dynamics of flow within the network, can be detrimental.

A comparison of the statistics for various bicycle-sharing programs[1] across the world (Table 1) shows an interesting (and possibly worrisome) trend: the rentals per bike day of the systems deployed in various cities in China are lower than their counterparts in Europe and Canada, despite having more stations and bicycles deployed. Although the lower number of rentals per bike day does not necessarily imply a lower utilization rate in terms of bike occupancy since one bike rental may involve a longer bike trip distance (or time) than another, this table does bring out an important issue—utilization of the bicycles—to BSS operators since it affects their financial viability because revenues from BSS are derived mainly from the utilization of the bicycles and membership fees. In this paper, we develop models that help predict the utilization rate of the bicycles.

The central focus of this paper is to understand how deployment and redistribution of bicycles in the network affect the utilization rate of the bicycles, and to what extent they affect the service level experienced by users. Although we restrict our discussion to bicycle-sharing systems, the ideas and techniques developed will be useful for the analysis of any system (e.g., car sharing) in which a fleet of vehicles is made available by a provider at the point of origin who cannot control trip destinations, as these are controlled by customers who arrive randomly during the planning horizon.

More specifically, we address the following pertinent issues in the management of a bicycle-sharing network:

• Given station locations, what is the appropriate number of bicycles to deploy in the network? The availability of bicycles affects the number of bicycle trips made and, in turn, the bicycle utilization rate. The former measures how much of the existing demand can be captured, whereas the latter affects the system's viability. Given the time-varying demand pattern, we need an optimal number of bicycles, appropriately located at the beginning of the day, to make effective use of the resources available to meet demand.

Table 1. Statistics for bicycle-sharing systems.

System operator	VELIB JCDecaux	BICING ClearChannel Adshel	BIXI Staonnement de Montréal	Beijing Fortune	Hangzhou Bicycle Service	Nanchang Xinfeida	Wuhan Xinfeida
City	Paris	Barcelona	Montréal	Beijing	Hangzhou	Nanchang	Wuhan
Start date	Jul-07	Mar-07	Spring 2009	Aug-05	May-08	Aug-09	Nov-08
Bicycles	20,600	6,000	5,000	10,000	50,000	1,000	20,000
Bike stations	1,451	400	400	1,000	2,000	30	718
Rentals/bike-day	12.5	16	15	2.32	8	4	5

• Impact of redistribution: Bicycle flow is dictated by customers' travel patterns. Some stations are departure dominant and others arrival dominant. For example, during peak morning hours, customer flows are almost always one directional (e.g., from home or interchange to office). This leads to bicycle stations that are full and others that are completely void of bicycles. It is thus necessary to constantly redistribute bicycles across the stations in the system to ensure that users will be able to find a dock where they can check out or return a bicycle. Our results show, interestingly, that the effectiveness of the redistribution strategy (measured by the number of additional trips supported) is in fact intricately tied to demand usage patterns and the number of bicycles deployed in the system.

• Number of bicycle docks: To make the bicycle-sharing program implementable, we need to consider how many bicycle docks to install at each station so that customers can return their bicycles on arrival at the destination station. Clearly, the number of docks needed at each station depends on utilization of the bicycles and how customer flows are supported in the system, and whether periodic redistribution is used to match supply with demand. In fact, our results show that redistribution has the potential to reduce the number of docks needed at each station.

We propose in this paper a simple proportional network flow model to help address the above issues. This model was first studied by Sahni (1974) as a generalization of the traditional network flow model. Ahuja et al. (1999) used this to study a class of water resource management problems in Sardinia, Italy. The equal flow condition is critical, because the amount of potable water transported must be the same in each time period in the network. Similarly, this constraint arises naturally in processor-sharing networks; see Koene (1982) and Chinneck (1995) for applications in other environmental and energy systems.

We develop the theory and explain the reasoning for our model in the next section. To validate our findings, in §3, we use a set of customer data from a Singapore mass rapid transit system to develop the demand model for our BSS. By focusing on this segment of the market for short trips, and through comparison with extensive simulation results, we demonstrate that the proposed model can be used to approximate to a reasonable level of accuracy the flow of bicycles in the system. More importantly, we demonstrate that the model can be used to address several pertinent

managerial issues in BSS management. In §4, we discuss various extensions and generalizations to this basic network flow model, and conclude the paper in §5.
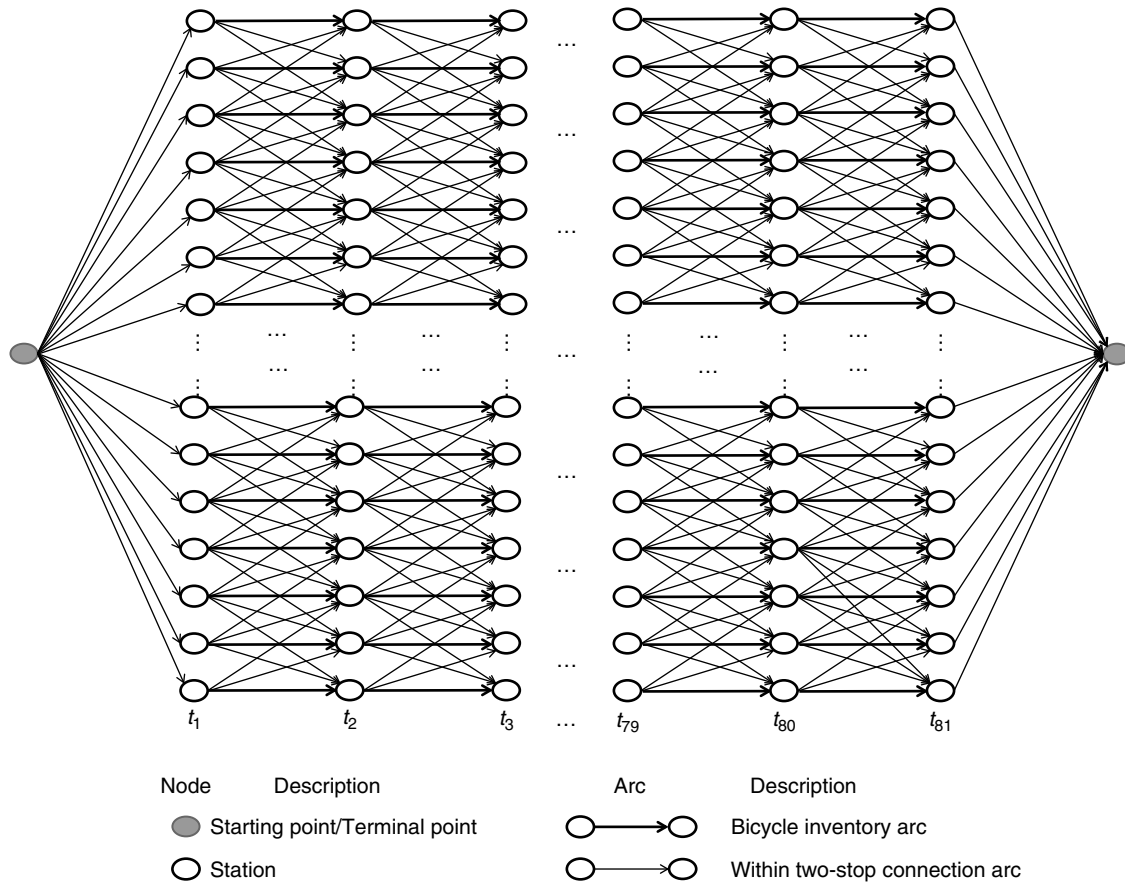
2. The Stochastic Network Flow Model

We assume that there is an initial allotment of bicycles at each train station. For each time period, passengers arrive randomly at the station to use the bicycles to travel to their destinations. The goal is to analyze and estimate the number of such trips that can be supported and substituted by the public bicycle-sharing system, based on the initial allotment of bicycles and the passenger arrival process. This is a technically challenging problem.

Formally, let $\mathscr{S}$ denote the set of stations in the network. In each time period t, the number of passengers who arrive planning to travel from station i to station j follows a Poisson process, with rate $r_{ij}(t)$. The total number of passengers arriving to use bicycles at station i is thus given by a Poisson process with rate $\sum_{j \neq i} r_{ij}(t)$. Within each time period, let $D_{ij}(t)$ and $D_i(t)$ denote the number of arrivals traveling on each link and into each station, respectively. We assume that all rides can be completed within a single time period.

Note that bicycles are allocated to passengers on a first-come-first-serve basis, so that whenever the initial stock of bicycles at a station is depleted, the latecomers will not be able to ride to their destinations using bicycles, and such demands are considered lost. Figure 1 shows the time-expanded view of the entire network, where the flow on each arc depends on the realization of the number of passengers traveling from origin to destination across a time period, and the number of bicycles available at the station.

REMARK. Note that in Figure 1 we assume that customers arrive at the stations and can complete their trips in one time period using bicycles. When customers are not able to reach their destination station using bicycles within a single time period, we only need to slightly modify the time-expanded network to allow arcs to extend across multiple time periods. The same linear programming (LP) based approach can be used to model the flow of customers in the network. In the most general case, of course we need to use a queueing network based approach[2] to model the flow of bicycles in this system. However, the associated optimization problem becomes intractable using this approach, because of the time-varying nature of the travel patterns.

Figure 1. Bicycle-flow network: Time-expanded graph.

Node	Description	Arc	Description
⬤	Starting point/Terminal point	◯⟶◯	Bicycle inventory arc
◯	Station	◯⟶◯	Within two-stop connection arc

To gain better insight into this problem, we consider an initial allotment of bicycles $x_i(t)$ at station i in time period t. The number of bicycle trips that will materialize in time period t will be $\min(x_i(t), D_i(t))$. However, the number of bicycles flowing from i to j will depend on the order of passenger arrivals at station i, and is more complicated to track. For $0 < p < 1$, let $D_i(t)[p]$ denote the number of tagged passengers where each passenger is tagged with probability p on arrival. More formally, let $\{\eta_i(p)\}$ denote a sequence of independent Bernoulli r.v.s with mean p, then

$$D_i(t)[p] = \sum_{k=1}^{D_i(t)} \eta_k(p).$$

By the well-known Poisson thinning Lemma, $D_i(t)[p]$ is Poisson with rate $p \times (\sum_{j \neq i} r_{ij}(t))$. Let $p_{ij}(t) \equiv r_{ij}(t)/\sum_{k: k \neq i} r_{ik}(t)$. Hence

$$D_{ij}(t) \sim D_i(t)[p_{ij}(t)].$$

By a slight abuse of notation, for some number of bicycles $x_i(t)$, let

$$\min(x_i(t), D_i(t))[p] = \sum_{k=1}^{\min(x_i(t), D_i(t))} \eta_k(p).$$

If there are $x_i(t)$ bicycles at station i, the number of bicycles leaving station i at time t is clearly $\min(x_i(t), D_i(t))$. The number of bicycles traveling from i to j, however, depends on the order of arrival of customers traveling to different destinations. In particular, the number of passengers traveling to j follows the distribution of

$$\min(D_i(t), x_i(t))[p_{ij}(t)].$$

The number of bicycles at station i at the end of the time period is given by

$$
\begin{aligned}
x_i(t+1) &= x_i(t) - \underbrace{\min(D_i(t), x_i(t))}_{\text{total departures}} \\
&\quad + \underbrace{\sum_{j: j \neq i} (\min(D_j(t), x_j(t))[p_{ji}(t)])}_{\text{total arrivals}} \\
&= x_i(t) - \sum_{j: j \neq i} (\min(D_i(t), x_i(t))[p_{ij}(t)]) \\
&\quad + \sum_{j: j \neq i} (\min(D_j(t), x_j(t))[p_{ji}(t)]).
\end{aligned}
\tag{1}
$$

The expected number of trips traversed using bicycles is given by

$$\sum_{t=0}^{N} \sum_{i \in \mathscr{S}} \sum_{j: j \neq i} E(\min(D_i(t), x_i(t))[p_{ij}(t)]).$$

Let $y_i(t) = E(x_i(t))$, and

$$
\begin{cases}
y_{ij}(t) = E(\min(D_i(t), x_i(t))[p_{ij}(t)]), \\
y_{ii}(t) = y_i(t) - \sum_{j: j \neq i} y_{ij}(t).
\end{cases}
$$

By the above definition, $y_{ij}(t)$ stands for the expected number of bicycles traveling from station i to station j during time period t. We next describe some simple structural properties of $y_{ij}(t)$.

LEMMA 1. $y_{ij}(t)/y_{il}(t) = r_{ij}(t)/r_{il}(t)$.

PROOF. We have

$$
y_{ij}(t) = E(\min(D_i(t), x_i(t))[p_{ij}(t)])
$$
$$
= E\left(\sum_{k=1}^{\min(x_i(t), D_i(t))} \eta_k(p_{ij}(t)) \right).
$$

Similarly,

$$
y_{il}(t) = E(\min(D_i(t), x_i(t))[p_{il}(t)])
$$
$$
= E\left(\sum_{k=1}^{\min(x_i(t), D_i(t))} \eta_k(p_{il}(t)) \right).
$$

Conditional on $\min(x_i(t), D_i(t))$, and using the fact that

$$
E(\eta_k(p_{ij}(t))) = p_{ij}(t), \quad \text{and} \quad E(\eta_k(p_{il}(t))) = p_{il}(t),
$$

we have

$$
y_{ij}(t)/y_{il}(t) = r_{ij}(t)/r_{il}(t). \quad \square
$$

Note that

LEMMA 2. $y_{ij}(t) \leqslant r_{ij}(t)$.

PROOF. This follows from

$$
y_{ij}(t) = E(\min(D_i(t), x_i(t))[p_{ij}(t)]) \leqslant E(D_i(t)[p_{ij}(t)])
$$
$$
= r_{ij}(t). \quad \square
$$

LEMMA 3. $y_i(t+1) = y_i(t) - \sum_{j: j \neq i} y_{ij}(t) + \sum_{j: j \neq i} y_{ji}(t)$.

PROOF. This follows from the flow conservation constraints in (1), i.e., for station i, the expected number of bicycles available at the beginning of period $t+1$ equals the expected number of bicycles at the beginning of period t, plus the net flow of bicycles into the station during period t. $\square$

Let Z^* denote the optimal objective value to the following linear programming problem:

$$
Z^* = \max \left(\sum_{t=0}^{N} \sum_{i \in \mathcal{S}} \sum_{j: j \neq i} y_{ij}(t) \right)
$$

subject to $y_i(t+1) = y_i(t) - \sum_{j: j \neq i} y_{ij}(t)$
$$
+ \sum_{j: j \neq i} y_{ji}(t), \quad \forall i, t;
$$
$$
y_i(t) = y_{ii}(t) + \sum_{j: j \neq i} y_{ij}(t), \quad \forall i, t;
$$
$$
\frac{y_{ij}(t)}{y_{il}(t)} = \frac{r_{ij}(t)}{r_{il}(t)}, \quad \forall i, j, l, t;
$$
$$
y_i(0) = x_i(0), \quad \forall i;
$$
$$
0 \leqslant y_{ij}(t) \leqslant r_{ij}(t), \quad \forall t, i \neq j.
$$

The variables $y_{ii}(t)$ denote the bicycles remaining in station i throughout time period t. The first constraint demonstrates that the number of bicycles available at the beginning of period $t+1$ equals the number of bicycles that remain at station i and the number of bicycles that arrive at station i during period t. Given an initial allotment of bicycles at station i, denoted by $x_i(0)$, the mean number of bicycle trips supported in the BSS on each link is a feasible solution to the above LP. Hence, we have the following:

THEOREM 1. Z^* denotes an upper bound to the expected number of bicycle trips in the system when the initial allotment of bicycles to station i is given by $x_i(0)$.

The above LP is surprisingly effective in providing a simple estimate of the performance (based on the number of bicycle trips the system can support) of the BSS with an initial bicycle inventory position $x_i(0)$. We will use this model extensively in the next section to examine bicycle utilization and the value of bicycle redistribution, using real transit data.

EXAMPLE. To see that the above LP is not exact, consider a three-station example in which there are two bicycles at station 3 initially, and none at the other two stations. Suppose $r_{31}(0) = r_{32}(0) = 1$, $r_{23}(t) = r_{32}(t) = 1$ for all $t > 1$, and $r_{ij}(t) = 0$ otherwise. If a bicycle flows from station 3 to station 1 in period $t = 0$, then it is stuck there throughout the rest of the day. In this case, to support the maximum flow in the network, the optimal LP solution suppresses the flow of bicycles from station 3 to stations 1 and 2 in the first period, maintaining the proportionality constraint, so that two bicycles will remain in station 3 from period 1 onward to serve the flow between station 2 and station 3. This LP solution dominates the expected number of trips in the stochastic network flow model.

Limitation. The previous example highlights a weakness in the LP approach: In real deployment, trips will be taken by riders and bicycles will not be held. The LP solution will only hold back bicycles in time period t if $r_{ij}(t)$ changes abruptly over the different time periods. In reality, we expect the change in $r_{ij}(t)$ to be gradual. Nonetheless, this also suggests a way to improve the accuracy of the

LP model to predict the expected flow in the system: by explicitly modeling the constraint that

$$x_i(t) - \underbrace{\min(D_i(t), x_i(t))}_{\text{total departures}} > 0$$

if and only if all passengers arriving in time t to station i can find a bicycle. Thus

$$\left(x_i(t) - \sum_{j: j \neq i} (\min(D_i(t), x_i(t))[p_{ij}(t)])\right)$$
$$\cdot (D_{ik}(t) - (\min(D_i(t), x_i(t))[p_{ik}(t)])) = 0$$

for every origin-destination pair i, k in all realizations of the stochastic system. This constraint could be handled by lifting the LP to a higher dimensional conic program to account for the quadratic terms in the constraint (cf. the approach using copositive cone in Natarajan et al. 2011). Note that this is not equivalent to forcing $y_{ii}(t)$ to zero unless $y_{ik}(t) = r_{ik}(t)$, since the constraint holds on all sample paths but may not hold for the product of $E[x_i(t) - \sum_{j: j \neq i} \min(D_i(t), x_i(t))[p_{ij}(t)]]$ and $E[D_{ik}(t) - \min(D_i(t), x_i(t))[p_{ik}(t)]]$. Indeed, our formulation allows for $y_{ik}(t) < r_{ik}(t)$ and yet $y_{ii}(t) > 0$.

2.1. Equilibrium State in Time-Invariant System

In the rest of this section, we further analyze the properties of this formulation (simple network flow with proportionality constraints) to gain insight on the problem.

Suppose the Poisson arrival in each time period is stationary with rate r_{ij}. Is there a way to characterize the number of bicycles in the equilibrium state of the bicycle-sharing network? We modify the LP to provide a glimpse of the answer to this problem.

In the equilibrium state, we expect $y_i(t+1) = y_i(t)$ as $t \to \infty$. Let

$$y_{ij} = \lim_{t \to \infty} y_{ij}(t).$$

The total number of bicycles in the system is denoted by N. Let y_{ij}^* denote the optimal solution to the following LP:

$$Z^*(\infty) = \max \sum_{i, j \in \mathscr{S}: j \neq i} y_{ij}$$

$$\text{subject to } \sum_{j: j \neq i} y_{ij} = \sum_{j: j \neq i} y_{ji}, \quad \forall i;$$
$$\frac{y_{ij}}{y_{il}} = \frac{r_{ij}}{r_{il}}, \quad \forall i, j, l;$$
$$0 \leq y_{ij} \leq r_{ij}, \quad \forall i, j;$$
$$\sum_i \left(y_{ii} + \sum_{j: j \neq i} y_{ij}\right) = N.$$

It can be seen easily that there exists i^* such that $y_{i^*j}^* = r_{i^*j}$ for all $j \neq i^*$; otherwise, we could scale the solution to improve the objective value. We call such nodes the *sink*

stations. Furthermore, if there exists i such that $y_{ii}^* > 0$ but $y_{ij}^* < r_{ij}$ for all $j \neq i$, then we could modify the solution by shifting y_{ii}^* to the station i^*, without affecting the feasibility and quality of the solution, i.e.,

$$y_{ii}^* \leftarrow 0, \quad y_{i^*i^*}^* \leftarrow y_{i^*i^*}^* + y_{ii}^*.$$

We call such nodes in which $y_{ij}^* < r_{ij}$ the *transient* stations. Note that WLOG, we can assume that $y_{ii}^* = 0$ when i is transient.

Let $z_i^* = \sum_{j: j \neq i} y_{ij}^*$. By the proportionality constraints, it is easy to see that

$$y_{ij}^* = \frac{r_{ij}}{\sum_{k: k \neq i} r_{ik}} z_i^*.$$

Note that z_i^* is a solution to the following system of linear equations:

$$z_i = \sum_{j: j \neq i} \left(\frac{r_{ji}}{\sum_{k: k \neq j} r_{jk}}\right) z_j, \quad i = 1, \dots, n. \tag{2}$$
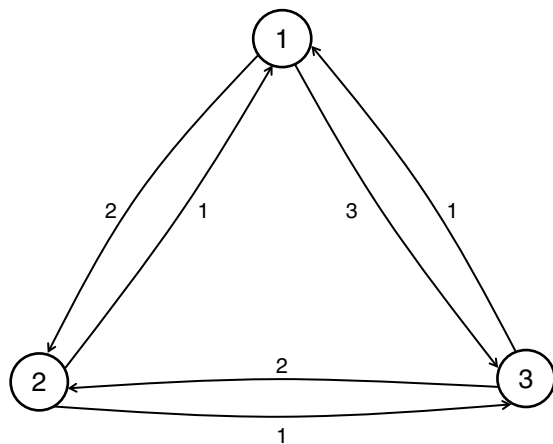
If the transition probability matrix constructed using $r_{ji}/(\sum_{k: k \neq j} r_{jk})$ is irreducible, then the above system of equations has a solution scaled to a constant. Note that $z_i^* \leq \sum_{k: k \neq i} r_{ik}$, since $y_{ij}^* \leq r_{ij}$ and $\sum_i z_i^* \leq N$. Since our objective is to maximize $\sum_i z_i^*$, the solution to the linear system (2) is scaled in such a way that either (i) $\exists \mathscr{S}$ such that $z_i^* = \sum_{k: k \neq i} r_{ik}$ for all $i \in \mathscr{S}$, and $z_i^* < \sum_{k: k \neq i} r_{ik}$ otherwise, or (ii) $\sum_i z_i^* = N$ and $z_i^* < \sum_{k: k \neq i} r_{ik}$ for all i. $\mathscr{S}$ corresponds to the set of sink nodes in the system. In case (i), the surplus $N - \sum_i z_i^*$ can be distributed to any of the y_{ii}^* variables for $i \in \mathscr{S}$ without affecting the optimality of the solution.

THEOREM 2. *The linear program $Z^*(\infty)$ may have multiple optimal solutions, but the flow solution y_{ij}^*, $i \neq j$, is uniquely determined by the rates r_{ij} if the transition probability matrix is irreducible. The "surplus" denoted by y_{ii}^* for the sink nodes are, however, nondetermined and can be distributed across different sink nodes.*

Since the surplus y_{ii}^* have zero weights in the objective function, having a large surplus does not improve the quality of the solution. This result indicates that given the rates r_{ij}'s, there is a limit N^* such that any number of bicycles beyond this limit N^* will not improve the performance of the system.

EXAMPLE. In Figure 2, we have three stations that are connected to each other. The number beside each direct arc (i, j) stands for the stationary arrival rate r_{ij}. Station 1 has a net outflow of three passengers per unit time, whereas stations 2 and 3 have net inflow of two and one passenger, respectively. We expect the average number of bicycles at station 1 to drain down to zero quickly, with the bulk of bicycles building up at stations 2 and 3. However, note that once the bicycles at station 1 drain down to zero, stations 2 and 3 immediately receive less inflow, and station 3 will now have a net outflow of two bicycles per unit time.

We use the outputs from the simulation model to plot the time-average level of bicycles (y_{ii}^*) at each station over

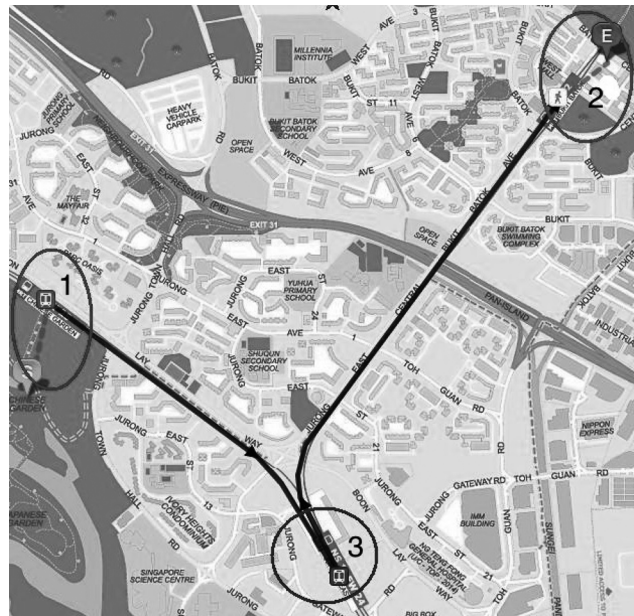**Figure 2.** Numerical example with three stations.

Figure 3. Routing from station Chinese Garden (1) to Bukit Batok (2) via Jurong East interchange (3) in Singapore.

2,000 periods and summarize the computational results in Table 2. In particular, the gap is calculated as 100% × (output of the deterministic model − output of the simulation model (average of 2,000 simulations))/output of the simulation model. We observe that the time-average number of bicycles at each station stabilizes after 10 time periods, and the LP model yields highly accurate predictions of the time-average level of bicycles in the stochastic system.

3. Bicycle-Sharing Demand from Short Train Rides

We test the performance of the proposed bicycle deployment model using a set of public transport data provided by the Singapore Land Transport Authority (LTA). We were given access to one week's worth of public transit data from Monday to Friday (April 2011), including data on 17 million transactional bus rides and 30 million mass rapid transit (MRT) transactions. For the computation study, we focus on train-ride data for the Singapore MRT System (SMRT).

Launched in 1987, the SMRT has grown from only one line section with five stations to a network of four lines (east–west, north–south, north–east, and the newly launched Circle Line), with 102 stations and 148.9 km of track. It operates from 5:00 A.M. to 01:00 A.M. each day, with peak morning traffic from 7:30 A.M. to 9:30 A.M. and peak evening traffic from 5:30 P.M. to 7:30 P.M. The longest trip in our data set can take up to 33 stops, but the average number of stops traversed is only around 7.7. Note that except for a handful of stations, commute time between neighboring stations is roughly two to three minutes. About 16% of the trips are short, i.e., passengers leave the train system within two stops of their starting station.

The statistics thus show that a significant proportion of train passengers commute for, at most, six minutes daily. Given that BSSs appeal mainly to short-distance commuters, short SMRT rides form a latent demand segment for the bicycle-sharing system. We use this set of train ridership data to construct the demand profile for the bicycle-sharing model proposed here. For instance, Figure 3 shows a scenario in which all trips from station 1 to station 2 must pass through the interchange (station 3), even though the two stations are only three kilometers apart. Note that station 1 and station 2 are within two stops of each other, with a transfer at the interchange. This is the scenario in which a BSS with docks at stations 1 and 2 will have the most appeal. For ease of exposition, we assume that all rides between the two stations in our data set will be transferred to the bicycle-sharing system, forming the time-varying demand rate for our BSS. Note that in practice, the substitution rate will be much lower; informal small-scale surveys indicate that the substitution rate could be as low as 3%, depending on time of day and weather conditions.

Our study will focus on origin-destination demands within two stops with or without transfer at an interchange in the transit network. An alternate public transport system, such as a public BSS located at MRT stations, is an attractive alternate for such commuters, especially during morning and evening peak hours. The challenge, however, is to determine the right level of bicycles to deploy at each station and how utilization rates are affected by demand patterns.

Next, we compare the proposed proportional network flow model with a simulation model to identify the operational characteristics of the BSS.

3.1. Bicycle Deployment and Utilization

We split the horizon into 15-minute intervals, starting from 05:00 A.M., to collect customer data on those alighting within two stations. There are 80 time intervals for each

Table 2. Computational results of the numerical example with three stations.

	Simulation				
	$t = 0$	$t = 10$	$t = 50$	Deterministic	Gap %
Avg no. of bicycles at station 1	5	1.656900000	1.656600000	1.65666651648	0.004015241
Avg no. of bicycles at station 2	2	5.506100000	5.506000000	5.50643297152	0.007863631
Avg no. of bicycles at station 3	3	2.837000000	2.837400000	2.83690051200	−0.017603722

day and 560 time intervals for each week. We use a directed time-expanded network to model each MRT station at each time interval on each day. Let $\mathcal{A}$ denote the arc set in the time-expanded network. There are two types of arcs in $\mathcal{A}$. The first is the one that links station i in time t to station j in time $t+1$, for all t in which station j is within two stops from station i. The other arcs are inventory arcs that join the same station across two consecutive time periods. Note that in cases in which certain trips between i and j can last more than 15 minutes (e.g., 15–30 minutes), we could simply modify our model by replacing the arc from i in time t to j in time $t+1$, with a new arc from i in time t to j in $t+2$, to model events in which trips last more than one time period.

We also adopt the following notations:

• We define the system bicycle utilization rate $\alpha(t)$ for each time period t as follows:

$$\alpha(t) \equiv \frac{\sum_{i,\, j:\, i \neq j} y_{ij}(t)}{\sum_i x_i(0)},$$

where $\sum_i x_i(0)$ represents the total number of bicycles positioned at all stations at the beginning of the planning horizon. Hence, $\alpha(t)$ is the proportion of bicycles in use at time t. Since the number of bicycles in the system is a constant,

$$\beta = \sum_t \alpha(t)$$

measures the total number of rides in the system divided by the total number of bicycles available, i.e., the (average) number of times each bicycle is being used.

Note that β determines the economic viability of the BSS: Each bicycle needs to be used more than a threshold value within a stipulated number of years to justify initial investment in the bicycle. In terms of revenues, however, we need to deploy a large number of bicycles to support a larger number of trips—yet this will decrease the average bicycle utilization rate β. Hence, we need to delicately balance the revenues generated (measured by the number of trips) with the utilization rate of the bicycles in the system.

With the above-defined notations, we can modify the LP developed in the earlier section to account for the (desired) utilization rate of the bicycles:

$$Z^*(\beta) = \max_{x_i(0),\, y_{ij}(t)} \left(\sum_{t=0}^{N} \sum_{i \in \mathcal{S}} \sum_{j:\, j \neq i} y_{ij}(t) \right)$$

subject to
$$y_i(t+1) = y_i(t) - \sum_{j:\, j \neq i} y_{ij}(t)$$
$$+ \sum_{j:\, j \neq i} y_{ji}(t), \quad \forall i, t; \quad (3)$$

$$\sum_{t=0}^{N} \sum_{i \in \mathcal{S}} \sum_{j:\, j \neq i} y_{ij}(t) \geqslant \beta \sum_i x_i(0); \quad (4)$$

$$y_i(t) = y_{ii}(t) + \sum_{j:\, j \neq i} y_{ij}(t), \quad \forall i, t; \quad (5)$$

$$\frac{y_{ij}(t)}{y_{il}(t)} = \frac{r_{ij}(t)}{r_{il}(t)}, \quad \forall i, j, l, t; \quad (6)$$

$$y_i(0) = x_i(0), \quad \forall i; \quad (7)$$

$$0 \leqslant y_{ij}(t) \leqslant r_{ij}(t), \quad \forall t, i \neq j. \quad (8)$$

Note that constraint (4) requires the weekly bicycle utilization rate to be at least β. The above LP determines the total number of bicycles and their deployment at the beginning (i.e., $x_i(0)$) of the planning horizon, to attain the desired utilization rate of β for the system. We solve the above model using the CPLEX LP solver to obtain the maximum number of substituted trips using bicycles, the number of bicycles to be optimally positioned at each station initially, and bicycle utilization rate $\alpha(t)$ at each time period.

We also compare the solutions obtained from the deterministic model with a simulation model. The detailed steps in implementing the simulation are given as follows.

• We fix β, solve the deterministic model outlined in the previous section, and obtain the optimal $x_i(0)$ to be deployed at each station i at the beginning of the planning horizon.

• We use $x_i(0)$ as the input to run the simulation model for a stochastic network flow system with Poisson demand at each arc in the network. We run the simulation 100 times for each β to obtain the sample average of the system performance.

• In each simulation, we use the direct time-expanded network and assume the number of customers arriving at each station during each 15 minutes time interval follows a Poisson process. In particular, the mean of the interarrival time for customers arriving at station i with destination station j at time index t equals to $15/r_{ij}(t)$. We then sort the customers at each node according to their arrival time at node i and discard those arrivals after 15 minutes. The bicycles at station i are used by the customers on a first-come-first-serve basis. We run this simulation for one week to obtain the number of bicycle trips supported and the bicycle utilization rate.
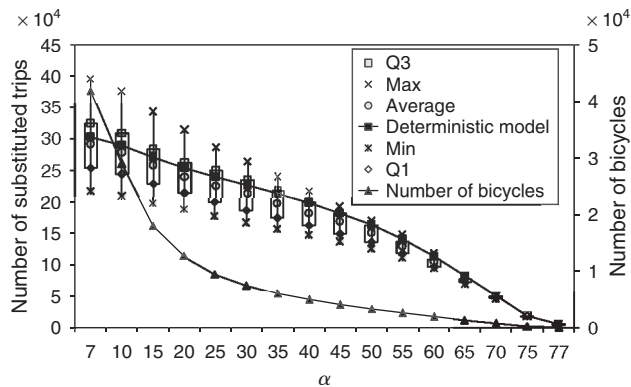
Figure 4. Short-trip substitution boxplot.

Figure 5. Expected number of bicycles in Jurong East Station, at different times in the day.

Figure 4 shows the performance of the bicycle-sharing network when we assume that all short trips (within two stops) in the train system are moved to the BSS. In the figure, the *x*-axis corresponds to the average daily utilization rate (denoted by α, where $\alpha = \beta/7$). The *y*-axis on the left shows the number of trips using bicycles, and the *y*-axis on the right shows the number of bicycles deployed in the system. The box plots (obtained via simulation) show that variations in the number of bicycle trips increase when the daily utilization rate decreases. More importantly, numerical results show that the deterministic LP model yields good approximation to the average number of bicycle trips supported in the stochastic network flow model.

The relationship between the number of bicycle trips supported by the system and the daily utilization rate appears to be almost linear when the daily utilization rate α is below 40: The number of bicycle trips decreases linearly as the targeted utilization rate of the system in this region increases. However, the number of bicycles in optimal deployment decreases drastically as the targeted daily utilization rate increases, when α is below 40.

These trade-offs have important implications. First, it appears that for this test case, an appropriate targeted utilization rate to operate successfully is $\alpha = 30 - 40$. At any rate below this level, we will need to deploy significantly more bicycles to support a small increase in the number of bicycle trips. In this range, however, the service level will not be high, as a significant portion of demand for rides cannot be supported. The average total demand within the system is around 308,000 trips, but when the system is operating at $\alpha = 40$, it can only support, on average, 182,035 trips (roughly 59% of demand).

Figure 5 shows the expected number of bicycles in one of the train stations (Jurong East, an interchange station in the MRT network) over the course of one day (80 time intervals on the *x*-axis). Commutes into the station are normally for short distances from neighboring train stations, to board trains going into city areas. Hence, there is a net flow of bicycles into the station during morning peak hours. Stock is only depleted during evening peak hours, when customers return to ride bicycles back to their residential
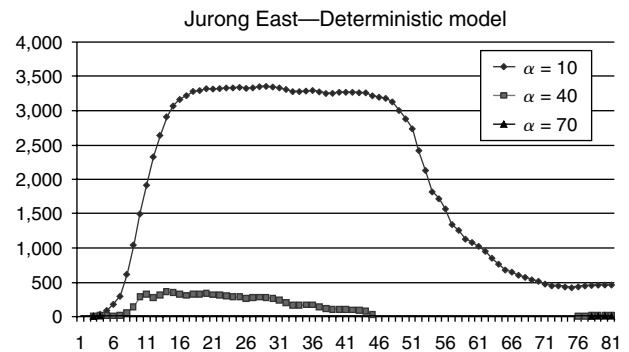
areas. The break in the curve (for $\alpha = 40$ or 70) shows time zones where no bicycle is expected to be at the station, i.e., all bicycles going into that station are immediately utilized, and hence the likelihood of unsatisfied demand for bicycles is high in that instance. Most demands can be satisfied when α is 10 (i.e., when a large number of bicycles are deployed), but the station will be empty most of the time, for $\alpha = 40$ or 70 (when the number of bicycles deployed is moderate or small). This shows the underlying difficulty in making a fee-charging BSS financially viable: A satisfactory service level can only be attained at low utilization rates, because of customers' travel timings and usage.

The stocking level in each station also fluctuates over time. Figure 6 shows the fluctuation of expected number of bicycles in three adjacent stations for $\alpha = 10$ (large number of bicycles deployed) and $\alpha = 70$ (small number of bicycles deployed), as obtained from the LP model. Ang Mo Kio station is at the middle and is about two km away from the Yio Chu Kang and Bishan stations, respectively. The figure shows that bicycle inventory in Yio Chu Kang had gone up by late morning in both scenarios, even though the initial number of bicycles at the beginning of the day was small in both cases. There is thus a high net inflow into this station, for the given initial deployment of bicycles. On the other hand, the Ang Mo Kio station had a large number of bicycles at the beginning in the high α case, but got depleted quickly over the day. In the low α case, the inventory remained low until the end of the day, when inventory started to increase. Both scenarios point out clearly that there is a need to move some bicycles from Yio Chu Kang to Ang Mo Kio during the morning periods, perhaps followed by some movement of bicycles from the Ang Mo Kio to the Bishan station later in the day. This can be achieved by redistributing the bicycles using dedicated trucks, or by introducing incentive schemes to encourage more bicycle travel along these routes at appropriate times.

The estimated number of bicycles obtained from the LP model can be a valuable aid for operators to pinpoint these critical routes for further enhancement of BSS operations. In the next section, we will exploit this observation to examine the impact of redistribution on the performance of the system.

Figure 6. Expected number of bicycles in three adjacent stations, at different times in the day.
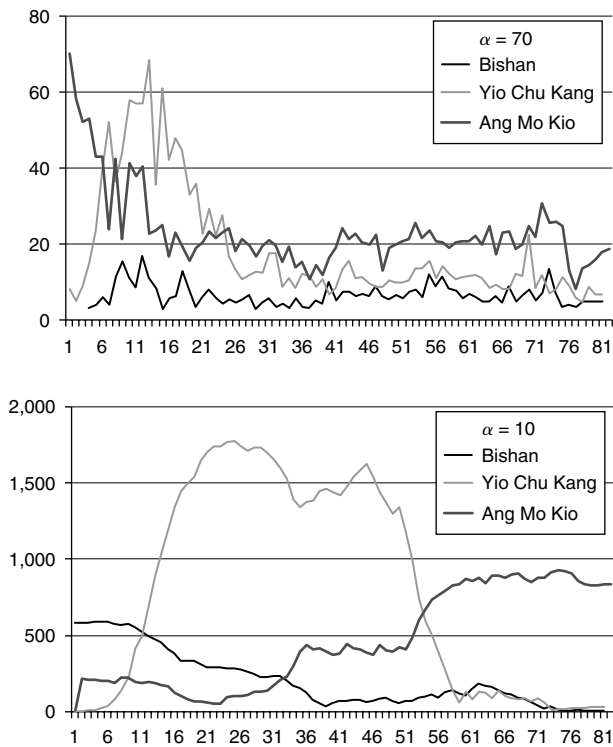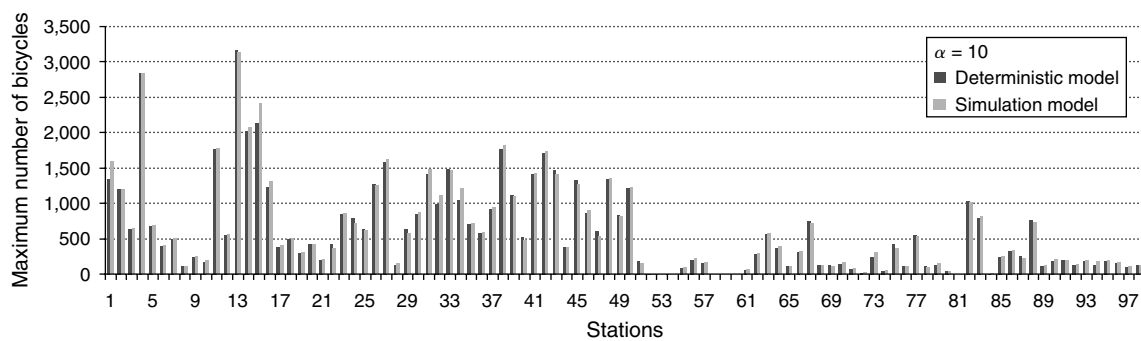

3.2. Number of Bicycle Docks Needed

Technically, we need to set up enough bicycle docks at each station so that customers will be able to return the bicycles when they reach their destinations. We calculate the number of docks needed for each station as the *maximum* bicycle quantity at each station across all time periods. Figures 7–9 compare the maximum bicycle quantity at each station during all time periods for $\alpha = 10$, 40, and 70 for the network flow and the simulation model.

Interestingly, data extracted from the network flow model are reasonably close to the actual peak inventory level at each station, as obtained from the simulation model. Furthermore, the number of docks needed to support storage of peak inventory decreases with increased utilization (i.e., fewer bicycles deployed).[3]

Computational results also show, interestingly, that with a smaller number of bicycles deployed ($\alpha = 70$), the system should deploy more bicycles near to and at stations in the central business district (stations 30–50 in the chart), leading to a relatively higher number of docks at these stations. With more bicycles available ($\alpha = 40$ or $\alpha = 10$), however, deployment of additional bicycles should be toward other congested areas, such as stations near the interchange in the east (stations 1–20), leading to a surge in the number of docks there.

This computational result suggests that as more bicycles are optimally deployed in the system, imbalances build up in different regions of the network and will require investment in docking capacity in different regions in the network.

3.3. Effectiveness of Bicycle Redistribution

With a slight abuse of notation, we redefine the time-expanded network to model customer flow for each day k. Let N_k denote the time index in the network on day k. We conduct the experiment as follows. We first solve the deterministic model $Z^*(\beta)$ proposed earlier, based on the one-week data, to obtain the number of bicycles deployed (denoted by C_β). We then use this as input to run the following program (P_k) for each day k:

$$Z_k^*(\beta) = \max_{x_i(0), y_{ij}(t)} \left(\sum_{t \in N_k} \sum_{i \in \mathscr{S}} \sum_{j: j \neq i} y_{ij}(t) \right)$$

$$\text{subject to} \quad y_i(t+1) = y_i(t) - \sum_{j: j \neq i} y_{ij}(t)$$
$$+ \sum_{j: j \neq i} y_{ji}(t), \quad \forall i, t;$$

$$\sum_i y_i(0) = C_\beta;$$

$$y_i(t) = y_{ii}(t) + \sum_{j: j \neq i} y_{ij}(t), \quad \forall i, t;$$

$$\frac{y_{ij}(t)}{y_{il}(t)} = \frac{r_{ij}(t)}{r_{il}(t)}, \quad \forall i, j, l, t;$$

$$0 \leqslant y_{ij}(t) \leqslant r_{ij}(t), \quad \forall i, j, t.$$

Figure 7. Number of docks: Deterministic model vs. simulation model ($\alpha = 10$).

Figure 8.   Number of docks: Deterministic model vs. simulation model ($\alpha = 40$).
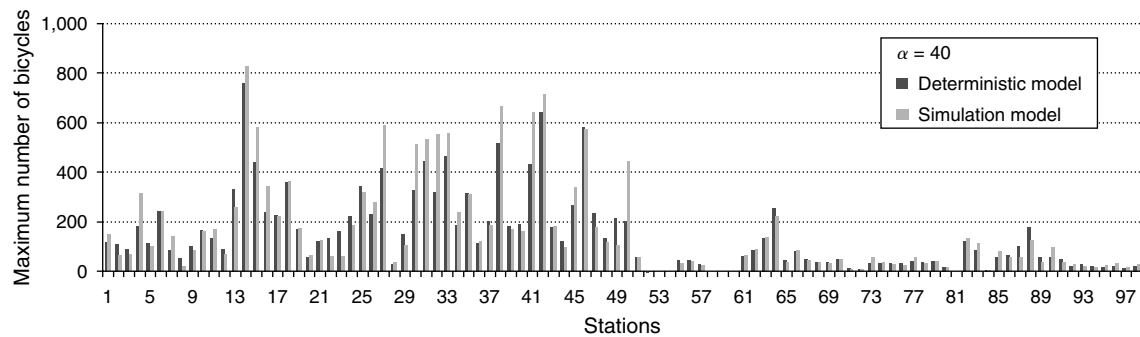
Figure 9.   Number of docks: Deterministic model vs. simulation model ($\alpha = 70$).
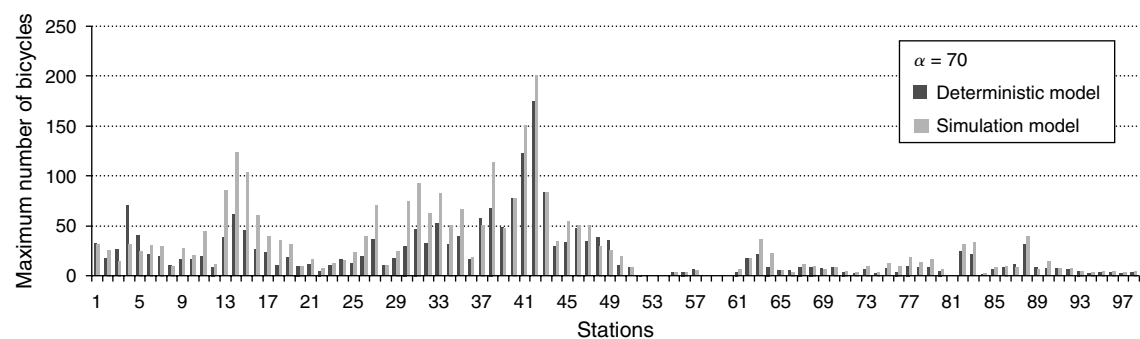
The above LP computes the optimal way to locate the C_β bicycles in the system, given the travel patterns of the day. Note that we solve an LP for each β. By implementing the above model for each day, we implicitly assume that the system performs redistribution at the end of each day.

This daily redistribution strategy affects performance of the BSS. Figure 10 shows that this strategy prevents surplus bicycles from building up at stations, and thus reduces the need to build more docks at each station. For $\alpha = 40$, it reduces the peak docking stations needed from 800 to around 700.[4]

Although a redistribution strategy can enhance system performance in terms of the number of substituted trips supported in the system and the number of docks needed at each station, it is a time-consuming and expensive task. The concern is, how often shall we perform the redistribution in the system? In the rest of this section, we run more experiments to examine the value of periodic redistribution. For ease of exposition, we assume that redistribution can be accomplished within a single time period. More generally, we can modify the time-expanded network to allow arcs to join multiple time periods if the time required for redistribution is more than one period. Figure 11 shows the trade-offs among the number of substituted trips supported, the total number of bicycles, and the number of periodic redistributions per day. In this set of experiments, we subdivide the time horizon evenly into 80 smaller time intervals per day, and perform periodic redistribution at equal time intervals over one day. For certain cases, when 80 time

intervals is not divisible by the number of redistributions per day, we keep the remainder in the last time interval of the day. Figure 11 shows the end result: when the total number of bicycles invested in the system is more than 30,000, frequent periodic redistribution does not add much to the number of bicycle trips supported by the system. Furthermore, a small number of daily redistributions (e.g., two to four) suffices, since more frequent redistribution will not add much to total supported bicycle trips.

4. Extensions and Discussions

The model developed in the previous section ignores two pertinent issues in BSS design: the impact of limited docks on flow in the system, and that the number of redistribution arcs per period is limited because of resources available (i.e., number of trucks used for redistribution). In this section, we discuss how these features can be handled by suitable reformulation of the model.

4.1. Resource-Constrained Redistribution Design

In reality, redistribution activities per time period are the key cost driver of most BSS operations, constituting close to 30% in most European cities. The number of redistribution routes per time period is usually constrained by Q, the number of trucks available. Also, suppose the maximum budget for bicycle investment in the system is C public bicycles. To incorporate these, we need to modify our LP

Figure 10. Number of docks: Deterministic model vs. simulation model with redistribution.
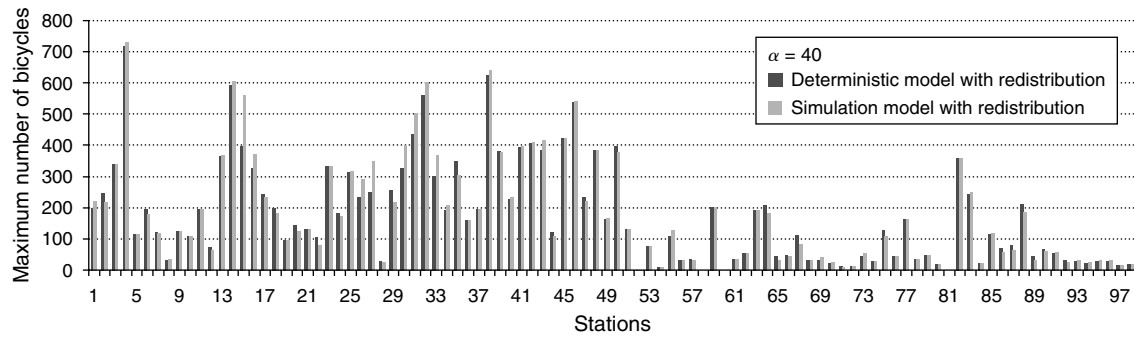
to bring in new decision variables $z_{ij}(t)$ to model the redistribution flows on arc (i, j) in time t. Let c_{ij} denote the cost of moving one bicycle from i to j, normalized to the revenue generated per ride supported in the system. For ease of exposition, we assume that redistribution can be completed within one time period; otherwise, we add the arc across multiple time periods to denote the time needed for redistribution. Our new model becomes

$$Z_Q^* = \max \left(\sum_{t=0}^{N} \sum_{i \in \mathscr{S}} \sum_{j: j \neq i} y_{ij}(t) \right) - \left(\sum_{t=0}^{N} \sum_{i \in \mathscr{S}} \sum_{j: j \neq i} c_{ij} z_{ij}(t) \right)$$

$$\text{subject to} \quad y_i(t+1) = y_i(t) - \sum_{j: j \neq i} (y_{ij}(t) + z_{ij}(t))$$
$$+ \sum_{j: j \neq i} (y_{ji}(t) + z_{ji}(t)), \quad \forall i, t;$$

$$y_i(t) = y_{ii}(t) + \sum_{j: j \neq i} (y_{ij}(t) + z_{ij}(t)), \quad \forall i, t;$$

$$\frac{y_{ij}(t)}{y_{il}(t)} = \frac{r_{ij}(t)}{r_{il}(t)}, \quad \forall i, j, l, t;$$

$$y_i(0) = x_i(0), \quad \forall i;$$

$$0 \leqslant y_{ij}(t) \leqslant r_{ij}(t), \quad \forall t, i \neq j;$$

$$\sum_i x_i(0) = C;$$

$$z_{ij}(t) \geqslant 0, \quad z_{i,j}(t) \leqslant M v_{i,j}(t), \quad \forall t, i \neq j;$$

$$\sum_{i,j} v_{i,j}(t) \leqslant \begin{cases} Q & \text{if } t \in \mathscr{T}, \\ 0 & \text{otherwise.} \end{cases}$$

In this formulation, M is a large constant, and $v_{i,j}(t)$ denotes the indicator that the arc (i, j) is used for redistribution in time t. The last constraint models the fact that redistribution is allowed on at most Q arcs, and in time period t when t is an allowable time period for redistribution (denoted by $\mathscr{T}$). We space out the redistribution periods in $\mathscr{T}$ so that we do not have to model explicitly that the redistribution arcs selected form a route for each truck.

Note that technically, the redistribution plan should be dynamic and deployed based on real-time information about the flow in the network. Hence, $z_{ij}(t)$ should be dynamically determined. However, since the constraints

Figure 11. 3-D illustration of periodic redistribution.
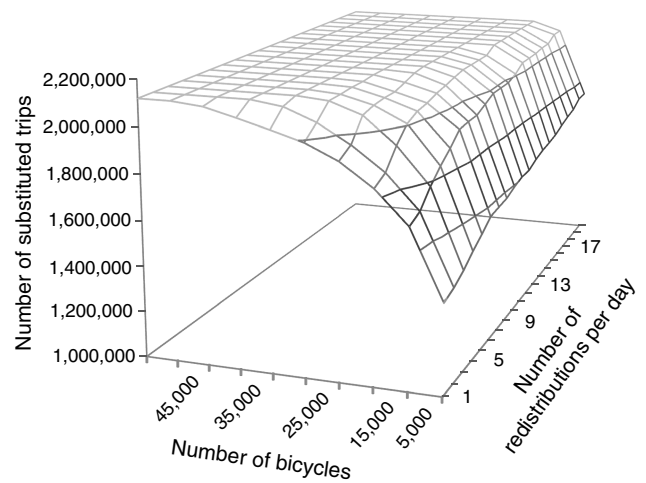
proposed above are satisfied in all sample paths, Z_Q^* provides an upper bound on the optimal dynamic redistribution solution, and $z_{ij}(t)$'s obtained above are estimated redistribution plans for future time periods. The above can be solved and implemented in a rolling-horizon format to update the plan for redistribution.

4.2. Modeling Dock Capacities

The availability of docks to receive returned bicycles is an important consideration in most BSS operations. We assumed an unlimited supply of docks in our formulation to simplify the dynamics of the flow of bicycles in the system. This assumption is reasonable for some BSSs such as "Call a Bike" in Germany or DATE BIKE at Sendai, Japan, where a bike does not need to be returned to a fixed rack. Instead, it can be returned somewhere close to a rental site, as long as the sensor can detect the returned bike. In this case, customers do not have the problem of finding an empty dock, and thus we may treat it as an uncapacitated station. Another situation where the uncapacitated station assumption is reasonable is when human agents are assigned to manage the bicycle stations; they can store overflow bicycles in temporary shacks, and thus mitigate the need to have the right levels of docking capacity

Figure 12. Mobile docking concept proposed by Isuda Singapore.

in the stations. This may be a costly option though, thus may not be feasible for some BSSs. A company in Singapore, Isuda, is attempting to address the docking issue with its new mobile dock concept (Figure 12) in which docks are on wheels and can be moved to other locations if necessary. Using this approach, we could improve bicycle utilization by periodically redistributing not only the bicycles but also the docks. Furthermore, docking sites could be easily relocated if necessary, therefore reducing the risk of installing docks at the wrong locations. This gives rise to an associated operational problem: how to manage mobile bicycles and docks. Our results should be a useful building block for any dynamic control policy for redistributing bicycles and docks within the network. Every redistribution plan has to balance the trade-off between taking docks and bicycles out of circulation in the current period, i.e., temporary loss of capacity, with the long-term expected gain of better positioning of bicycles and docking capacity within the network. The LP approach developed in this paper can be used to estimate the long-term expected gain for the current time period, given the starting bicycle and docking capacities at each station.

Another way to approach docking capacity is to introduce this into the dynamics of the flow of bicycles in the network. Suppose that each station i has a physical docking capacity of K_i that cannot be changed. The flow arriving at station i in time t cannot be more than K_i; otherwise, we assume that the IT system in the BSS is able to inform customers in advance and prevent an excessive inflow of bicycles into the station. We use new variables $y_{ij}^D(t)$ to denote the effective flow of bicycles after accounting for the effect of docking capacities. An upper bound to the number of trips supported in the network can be obtained by solving

$$Z_D^* = \max \left(\sum_{t=0}^{N} \sum_{i \in \mathscr{S}} \sum_{j:\,j \neq i} y_{ij}^D(t) \right)$$

subject to $y_i(t+1) = y_i(t) - \sum_{j:\,j \neq i} y_{ij}^D(t)$

$$+ \sum_{j:\,j \neq i} y_{ji}^D(t), \quad \forall i, t;$$

$$y_i(t) = y_{ii}^D(t) + \sum_{j:\,j \neq i} y_{ij}^D(t), \quad \forall i, t;$$

$$\frac{y_{ij}(t)}{y_{il}(t)} = \frac{r_{ij}(t)}{r_{il}(t)}, \quad \forall i, j, l, t;$$

$$\frac{y_{ij}^D(t)}{y_{kj}^D(t)} = \frac{y_{ij}(t)}{y_{kj}(t)}, \quad \forall i, j, k, t;$$

$$\sum_{i} y_{ij}^D(t) \leqslant K_j, \quad \forall j, t;$$

$$y_i(0) = x_i(0), \quad \forall i;$$

$$0 \leqslant y_{ij}^D(t) \leqslant y_{ij}(t) \leqslant r_{ij}(t), \quad \forall t, i \neq j.$$

The constraint

$$\frac{y_{ij}(t)}{y_{il}(t)} = \frac{r_{ij}(t)}{r_{il}(t)}$$

captures the proportionality condition for Poisson flow out of station i into station j and l, whereas the constraint

$$\frac{y_{ij}^D(t)}{y_{kj}^D(t)} = \frac{y_{ij}(t)}{y_{kj}(t)}$$

modifies the flow further by factoring in the docking capacity at station j, and enforces the proportionality constraint for Poisson flow into station j. Note that this gives rise to a nonlinear program. The constraint

$$\sum_{i} y_{ij}^D(t) \leqslant K_j$$

models the constraint that the docking capacity at j is fixed at K_j.

5. Concluding Remarks

In this paper, we propose a novel bicycle-sharing model in which customers use bicycles to substitute for their short-distance trips. We use a deterministic LP model to approximate the system performance of the stochastic system, and show that the deterministic model can imitate the actual system performance very closely, based on actual Singapore MRT ridership data. We conduct extensive numerical experiments to examine important issues such as the bicycle utilization rate, the value of bicycle redistribution, and the number of bicycle docks that should be set up at each station.

Our model can also be extended to incorporate the scenario of using bicycles to transport customers between MRT stations and neighborhoods, assuming that there is no spillover of demand from one location to another. We have implemented our model using a set of bus-transit data in a new town in Singapore, and identified the ideal locations to set up bicycle stations for the network. Our numerical results suggest that the optimal location choices are robust to input errors: for various demand scenarios, the same set of locations are identified as optimal. Furthermore, adding

proportionality significantly improves the estimation accuracy of the number of trips supported.

Another interesting direction for research will be to explore the usage of incentive schemes to balance flow. Our approach hinges crucially on the fact that system parameters $r_{ij}(t)$ are given as input. When they are endogenous to the model, i.e., when promotional activities can be used to influence the flow rate between i and j, then the problem is still unsolved. We leave these and other issues to future research.

Endnotes

1. See http://transport-solutions.blogspot.com/2010/08/revolutionizing-public-transport-with.html.
2. We thank Prof Gideon Weiss for pointing this out.
3. Note that we have assumed all customers will use bicycles to substitute their short-distance MRT trips (within two-Stop), upon the availability of the bicycles. We have thus actually obtained a gross overestimate on the total volume of trips that can be substituted by bicycles. In reality, only a small percentage of the short-distance commuters captured in the data will choose to use bicycles, say 10%. Therefore, all our numbers must be scaled down by a factor of 10 accordingly. In this case, we can see that for $\alpha = 40$, the maximum number of bicycle docks we need to set up among all stations is no more than 80 for our system.
4. If we assume that the take-up rate for bicycle trips is only 10% of the full demand, then the corresponding number of docks needed will be reduced by 90%, i.e., from 700 to 70 docks.

Acknowledgments

References

Ahuja R, Orlin J, Sechi G, Zuddas P (1999) Algorithms for the simple equal flow problem. *Management Sci.* 45(10):1440–1455.
Chinneck JW (1995) Processing network models of energy/environment systems. *Comput. Indust. Engrg.* 28(1):179–189.
DeMaio P (2003) Smart bicycles: Public transportation for the 21st century. *Transportation Quart.* 57(1):9–11.
DeMaio P (2009) Bicycle-sharing: History, impacts, models of provision, and future. *J. Public Transportation* 12(4):41–56.
DeMaio P, Gifford J (2004) Will smart bicycles succeed as public transportation in the United States? *J. Public Transportation* 7(2):1–15.
Koene J (1982) Minimal cost OW in processing networks: A primal approach. Ph.D. thesis, Technische Universiteit Eindhoven, Eindhoven, The Netherlands. http://alexandria.tue.nl/extra1/PRF4A/8203150.pdf.
Lathia N, Ahmed S, Capra L (2012) Measuring the impact of opening the London shared bicycle scheme to casual users. *Transportation Res. Part C* 22:88–102.
Lin J-R, Yang T-H (2011) Strategic design of public bicycle sharing systems with service level constraints. *Transportation Res. Part E* 47(2):284–294.
Lin J-R, Yang T-H, Chang Y-C (2013) A hub location inventory model for bicycle sharing system design: Formulation and solution. *Comput. Indust. Engrg.* 65(1):77–86.
Nair R, Miller-Hooks E (2011) Fleet management for vehicle sharing operations. *Transportation Sci.* 45(4):524–540.
Natarajan K, Teo CP, Zheng Z (2011) Mixed 0-1 linear programs under objective uncertainty: A completely positive representation. *Oper. Res.* 59(3):713–728.
OBIS Project (2011) Optimising bike sharing in European cities—A handbook. http://www.eltis.org/docs/tools/Obis_Handbook.pdf.
Raviv T, Kolka O (2013) Optimal inventory management of a bike-sharing station. *IIE Trans.* 45(10):1077–1093.
Raviv T, Tzur M, Forma IA (2012) Static repositioning in a bike-sharing system: Models and solution approaches. Working paper, Tel Aviv University, Tel Aviv, Israel.
Sahni S (1974) Computationally related problems. *SIAM J. Comput.* 3:262–279.
Schuijbroek J, Hampshire R, van Hoeve W-J (2013) Inventory rebalancing and vehicle routing in bike sharing systems. Working paper, Carnegie Mellon University, Pittsburgh.

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